Drastic fall-off of the thermal conductivity for disordered lattices in the limit of weak anharmonic interactions

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Abstract

We study the thermal conductivity, at fixed positive temperature, of a disordered lattice of harmonic oscillators, weakly coupled to each other through anharmonic potentials. The interaction is controlled by a small parameter $\epsilon > 0$. We rigorously show, in two slightly different setups, that the conductivity has a non-perturbative origin. This means that it decays to zero faster than any polynomial in ϵ as $\epsilon \to 0$. It is then argued that this result extends to a disordered chain studied by Dhar and Lebowitz [12], and to a classical spins chain recently investigated by Oganesyan, Pal and Huse [23].

1 Introduction

It is generally admitted that the thermal properties of solids can be derived from molecular dynamics, but it remains to this day a widely open conjecture, at least from a mathematical point of view [9][13][19]. Indeed, the few hamiltonian systems that can be handled analytically to a large extent, also appear to have a very pathological thermal behavior. So is it for the ordered harmonic chain [26], for the Toda lattice (see Section 6.3 in [7]), and for a one-dimensional system of colliding particles [28]. As we will see soon, disordered one-dimensional harmonic chains also fall into this category.

This said, much progress have been recently accomplished in the case of a harmonic crystal that is weakly perturbed by means of anharmonic interactions [15][2][20]. In particular, a Boltzmann-like equation for phonons can be, at least formally, derived in the limit where the interaction vanishes, after that time has been properly rescaled. A similar strategy is implemented in [13][19] to another kind of system. In these examples, the conductivity can thus be understood in a perturbative regime.

Besides anharmonicity, impurities constitute another possible mechanism destroying ballistic transport of energy. In disordered harmonic crystals, one assumes that the mass of each atom is random [27][11]. The dynamics is integrable and, in dimension one, the conductivity of disordered harmonic chains is by now rather well understood [29][1]. In particular, in presence of everywhere onsite pinning, Anderson localization of the eigenmodes forbids any transfer of energy [6]. Though still integrable in the mathematical sense, the dynamics is much less understood in higher dimensions. Normal conductivity is expected in dimension three if the disorder is small enough [10].

We here look at the effect of both disorder and anharmonic interactions on a lattice of pinned harmonic oscillators. Three studies have attracted our attention. First, Dhar and Lebowitz [12] conclude from numerical experiments that the conductivity of a one-dimensional pinned disordered harmonic chain becomes positive as soon as some anharmonicity is added, destroying thus localization. The same conclusion is then reached by Oganesyan, Pal and Huse [23], which study a chain of classical spins. Moreover, they observe a very rapid fall-off of the conductivity as the parameter $\epsilon > 0$ (called J in [23]) controlling the interaction goes to zero. Finally, Basko [3] argues that the conductivity in a comparable system can be understood as an effect of Arnold diffusion. He also gives an expression for the conductivity, predicting a decay with $\epsilon \to 0$ that is faster than any power law (equations (3.6) in [3] with ρ in place of ϵ).

Let us describe our contribution. We study a d-dimensional lattice of 1-dimensional harmonic oscillators with random eigenfrequencies, as described in detail in Section 2. For $\epsilon = 0$, they just evolve independently from each other, but, for $\epsilon > 0$, each oscillator is coupled to its neighbors through an anharmonic potential of strength ϵ.

Since the rigorous analysis of the conductivity of this system is likely out of reach, we study it in two slightly different simplified setups. Let $n >> 1$ be some large integer that we fix. The first simplifying procedure consists in following the dynamics only up to very large time-scales of order ϵ^{-n}, and study the conductivity in the strict weak coupling limit $\epsilon \to 0$, as in [2][20][13][19] for $n = 2$. Alternatively, we can perturb the hamiltonian dynamics by a stochastic noise that preserves the energy of individual

atoms, and possibly models some chaotic behavior of the system. This noise is however chosen such that it only becomes sensitive at time-scales of order ϵ^{-n}. It destroys hypothetical ballistic motion of energy only after such long times. These two approaches are very close in spirit.

Our results are stated in Section 3. Let $0 < \beta < +\infty$ be some given inverse temperature. Let $\kappa(\epsilon)$ be the Green-Kubo conductivity at inverse temperature β (see (3) below). Theorem 1 deals with the first setup. Let m be an integer such that $1 << m \le n$. It asserts that that $\epsilon^{-m}\kappa(\epsilon)$ vanishes in the weak coupling limit, no matter how large the integers $m \le n$ are taken. We believe this to be a strong indication that $\kappa(\epsilon) = \mathcal{O}(\epsilon^m)$ for every $m \ge 1$. Theorem 2 deals with the second setup, and asserts that, in that case, $\kappa(\epsilon) = \mathcal{O}(\epsilon^{n+2})$. It extends, in a simpler model, results obtained in [5][6], where the noise was dominating on the anharmonic interactions. Incidentally, a comparison of our work with [5] and [6] reveals that the effect of noise and that of anharmonic potentials can sometimes be very different in some respects.

Since the conductivity is defined at positive temperature, we find it hard to establish a clear connection between our work and the numerous existing results dealing with the possible dispersion of a finite energy packet at infinite volume (among many other references, let us mention [14][24][21] for mathematical results, or [22] for numerical ones). Still, in the perturbative regime that we consider, our results actually follow from a purely local analysis ; they are based on the lack of resonances in the system, a common phenomenon well described for example in Section 2 of [16].

Theorems 1 and 2 are proven in Section 4. In Section 5, we show, without providing a full mathematical treatment, how to adapt our line of reasoning to obtain similar conclusions for the disordered chain of [12] and the spin chain of [23], in agreement with the observations of [23] and the predictions of [3]. While the comparison with our model is completely straightforward in the case of the spin chain, if some assumption on the distribution of magnetic field is added, at least some technical work would be required to get a true mathematical proof for the chain of [12]. We finally suggest a connection with a chain of weakly coupled identical strongly anharmonic oscillators.

2 Model

Let $N \ge 3$, and let $\mathbb{Z}_N$ be the set of integers modulo N. Let also $d \ge 1$. We consider a set of one-dimensional classical oscillators, with equilibrium positions on the periodic d-dimensional lattice $\mathbb{Z}_N^d$. The phase space consists thus in the set of points

$$(q,p) \; = \; (q_x, p_x)_{x \in \mathbb{Z}_N^d} \; \in \; (\mathbb{R}^2)^{N^d}.$$

The Hamiltonian is written as

$$H(q,p) \; = \; \frac{1}{2} \sum_{x \in \mathbb{Z}_N^d} \left(p_x^2 + \omega_x^2 q_x^2 \right) + \epsilon \sum_{x \in \mathbb{Z}_N^d} U(q_x) + \frac{\epsilon}{2} \sum_{x \in \mathbb{Z}_N^d} \sum_{y : |x-y|_1 = 1} V(q_x - q_y)$$

with the two following definitions.

First, the eigenfrequencies of the decoupled harmonic oscillators, $\omega = (\omega_x)_{x \in \mathbb{Z}_N^d}$, form a sequence of independent and identically distributed random variables, with law $\mathsf{P}_\star$ independent of N. Let $\mathsf{E}_\star(\cdot)$ be the associated expectation. While more general distributions could likely be considered, we will assume that $\mathsf{P}_\star$ admits a bounded density, so that in particular $\mathsf{E}_\star(\omega_x^2) - \mathsf{E}_\star(\omega_x)^2 > 0$, and that, for some constants $0 < \omega_- < \omega_+ < +\infty$, the bound $\omega_- \leq \omega_x \leq \omega_+$ holds almost surely.

Second, the coupling constant ϵ is strictly positive. The potentials U and V belong to $\mathcal{C}_{temp}^\infty(\mathbb{R})$, the space of infinitely differentiable functions with polynomial growth. The pinning potential U is taken such that, for every $\alpha > 0$, for some constant $c > 0$, and for all $x \in \mathbb{R}$,

$$\int_{\mathbb{R}} e^{-\alpha(\omega_- y^2 + \epsilon U(y))} \, dy \; < \; +\infty \qquad \text{and} \qquad \omega_- + \epsilon \, U''(x) \; \geq \; c.$$

The interaction potential V is supposed to be symmetric, meaning that $V(-x) = V(x)$ for all $x \in \mathbb{R}$. Moreover, in order to ensure good decay properties of the Gibbs measure, we require the pinning to dominate over the interaction: similarly to (2.3) in [8], one asks that there exists a constant $\mathrm{C} < +\infty$ such that, for every $x, y \in \mathbb{R}$,

$$V''(x-y)^4 \; \leq \; \mathrm{C} \left(1 + U''(x)^2\right)\left(1 + U''(y)^2\right).$$

We possibly perturb the hamiltonian dynamics by a stochastic noise that preserves the energy of each oscillator. Let $\epsilon' \geq 0$. The full generator of the dynamics writes

$$L \; = \; A_{har} + \epsilon A_{anh} + \epsilon' S \; = \; A_{har} + \epsilon A_{anh}^{(0)} + \epsilon A_{anh}^{(1)} + \epsilon' S$$

with

$$
\begin{aligned}
A_{har} u \; &= \; \sum_{x \in \mathbb{Z}_N^d} \left(p_x \, \partial_{q_x} u - \omega_x^2 q_x \, \partial_{p_x} u\right), \\
A_{anh}^{(0)} u \; &= \; - \sum_{x \in \mathbb{Z}_N^d} U'(q_x) \, \partial_{p_x} u, \\
A_{anh}^{(1)} u \; &= \; - \sum_{x \in \mathbb{Z}_N^d} \sum_{y : |x-y|_1 = 1} V'(q_x - q_y) \, \partial_{p_x} u, \\
Su \; &= \; \sum_{x \in \mathbb{Z}_N^d} \left(u(\ldots, -p_x, \ldots) - u(\ldots, p_x, \ldots)\right).
\end{aligned}
$$

We denote by $\left(X_t^{(\epsilon, \epsilon')}\right)_{t \geq 0}$, or simply by $(X_t)_{t \geq 0}$, the Markov process on $(\mathbb{R}^2)^{N^d}$ generated by L.

Besides $\mathsf{E}_\star(\cdot)$, we will consider two other expectations. Let $0 < \beta < +\infty$ be an inverse temperature, that will be considered as a fixed parameter in the sequel. Let $\langle \cdot \rangle_\beta$ be the corresponding Gibbs measure, defined by (2) below. We will just denote by $\mathsf{E}(\cdot)$ the expectation over the realizations of the noise. We will write $\mathsf{E}_\beta(\cdot)$ for $\langle \mathsf{E}(\cdot) \rangle_\beta$.

We need some extra informations on the Gibbs measure. Its density ρ_β is given by

$$\rho_\beta(q, p) \; = \; e^{-\beta H(q,p)}/Z(\beta),$$

where $Z(\beta)$ is a normalization constant. This density factorizes as $\rho_\beta(q, p) = \rho'_\beta(q) \cdot \rho''_\beta(p)$, and the density ρ''_β factorizes again:

$$\rho''_\beta(p) = \left(\frac{\beta}{2\pi}\right)^{N^d/2} \prod_{x \in \mathbb{Z}_N^d} e^{-\beta p_x^2/2}.$$

While $\rho'_\beta(q)$ does not factorize, the conclusion of Theorem 3.2 in [8] holds thanks to our hypotheses on the hamiltonian H^1: there exist constants $C < +\infty$ and $c > 0$ such that, for any functions $u, v \in \mathcal{C}_{temp}^\infty(\mathbb{R}^{N^d})$ satisfying $\langle u \rangle_\beta = \langle v \rangle_\beta = 0$, it holds that

$$|\langle f \cdot g \rangle_\beta| \leq C e^{-cd(S(u),S(v))} \langle (\nabla_q u)^2 \rangle_\beta^{1/2} \langle (\nabla_q v)^2 \rangle_\beta^{1/2}. \tag{1}$$

Here, $S(u)$ is the support of a function u, defined as the smallest subset of $\mathbb{Z}_N^d$ such that u can be written as a function of the variables q_x for $x \in S(u)$, whereas $d(S(u), S(u))$ is the smallest distance between any point in $S(u)$ and any point in $S(v)$.

3 Conductivity

Let $x \in \mathbb{Z}_N^d$. The energy of particle x is given by

$$e_x = \frac{1}{2}(p_x^2 + \nu_x q_x^2) + \epsilon U(q_x) + \frac{\epsilon}{2} \sum_{y:|x-y|_1=1} V(q_x - q_y).$$

For $y \in \mathbb{Z}_N^d$ such that $|x - y|_1 = 1$, one defines the current

$$\epsilon j_{x,y} = \epsilon \frac{p_x + p_y}{2} V'(q_x - q_y).$$

Since V is symmetric, one has $j_{y,x} = -j_{x,y}$. With this definition, one computes that

$$L e_x = \epsilon \sum_{k=1}^d \left(j_{x-\mathsf{e}_k,x} - j_{x,x+\mathsf{e}_k} \right)$$

where e_k is the kth unit vector.

To simplify some further notations, we find it convenient to define now a set of oriented bonds in the lattice $\mathbb{Z}_N^d$:

$$\widetilde{\mathbb{Z}}_N^d = \left\{ (x, y) \in (\mathbb{Z}_N^d)^2 : y = x + \mathsf{e}_k \text{ for some } 1 \leq k \leq d \right\}.$$

The total current and the rescaled total current are defined by

$$\epsilon J_N = \epsilon \sum_{\widetilde{x} \in \widetilde{\mathbb{Z}}_N^d} j_{\widetilde{x}} \quad \text{and} \quad \epsilon \mathcal{J}_N = \frac{\epsilon}{N^{d/2}} J_N. \tag{2}$$

Then, if the limits exist, the Green-Kubo conductivity of the chain is defined by (see for example [18])

$$\kappa(\epsilon, \epsilon') = \lim_{t \to \infty} \lim_{N \to \infty} \kappa(\epsilon, \epsilon', N, t) = \lim_{t \to \infty} \lim_{N \to \infty} d^{-1}\beta^2 \, \mathsf{E}_\beta \left(\frac{\epsilon}{\sqrt{t}} \int_0^t \mathcal{J}_N \circ X_s^{(\epsilon,\epsilon')} \, ds \right)^2. \tag{3}$$

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[1] Strictly speaking, a slight adaptation of this theorem is needed to cover our needs: going through the proof, one checks that the randomness on the pinning does not affect the result, and that the constants can be taken independent of ϵ.

It maybe could look more natural to define the total current, and the associated Green-Kubo conductivity, with respect to a given direction, say e_k, for some $1 \leq k \leq d$. This means that the sum in (2) should be restricted to currents on bonds of the type $\widetilde{x} = (x, x + \mathsf{e}_k)$ for $x \in \mathbb{Z}_d^N$. Our results below still hold with such a restriction.

We first prove a result on the fall-off of the conductivity in a weak coupling limit ; it is physically most meaningful for $1 << m << n$:

Theorem 1. *Let $n \geq 1$ and let $\epsilon' = 0$. Let also $1 \leq m \leq n$. For almost all realizations of the eigenfrequencies, it holds that*

$$\lim_{t \to \infty} \limsup_{\epsilon \to 0} \limsup_{N \to \infty} \epsilon^{-m} \left\langle \left(\frac{\epsilon}{\sqrt{\epsilon^{-n}t}} \int_0^{\epsilon^{-n}t} \mathcal{J}_N \circ X_s^{(\epsilon, 0)} \, \mathrm{d}s \right)^2 \right\rangle_\beta = 0.$$

If the energy conserving noise is added, the following also holds:

Theorem 2. *Let $n \geq 1$ and assume that*

$$\epsilon' = \epsilon^n.$$

Then there exist constants $\mathrm{C} = \mathrm{C}(n) < +\infty$ and $\epsilon_0 > 0$ such that, for almost all realizations of the eigenfrequencies, and for all $\epsilon \in [0, \epsilon_0]$, one has

$$\limsup_{t \to \infty} \limsup_{N \to \infty} \mathsf{E}_\beta \left(\frac{\epsilon}{\sqrt{t}} \int_0^t \mathcal{J}_N \circ X_s^{(\epsilon, \epsilon')} \, \mathrm{d}s \right)^2 \leq \mathrm{C}(n) \cdot \epsilon^{n+2}.$$

Remarks. 1. The hypothesis that the law of the frequencies $(\omega_x)_{x \in \mathbb{Z}_N^d}$ admits a density cannot be completely dropped out for the proof to work. Assume indeed, at the opposite, that the frequencies should take only two values. In that case, among three oscillators, two at least have the same frequency. Then, as long as one takes $d = 1$ and $n > 1$, or $d \geq 2$, some fatal resonances appear and invalidate our line of reasoning.

2. It is possible that the weak coupling limit or the noise actually destroys some effects, such as a very slow ballistic transport, that become dominant when looking to the system at large enough time-scales. For example, let us assume that n is given, that $d = 1$, that all the interactions are harmonic, and that $(\omega_x)_{x \in \mathbb{Z}_N}$ form a sequence of $2n$ different frequencies, arranged periodically. One checks that our proof works in this particular case, although the true conductivity is known to be infinite. Since however, in our theorems, n can be taken arbitrarily large, such a scenario only becomes possible at time-scales larger than any inverse polynomial, and does not affect our main message that the transport of energy has a non-perturbative origin.

3. The proofs are achieved by gathering three different ideas. Given $\widetilde{x} \in \widetilde{\mathbb{Z}}_d^N$, we first approximately solve the Poisson equation $-Lu = j_{\widetilde{x}}$, assuming a non-resonance condition on the random frequencies of the oscillators located near $\widetilde{x}$, as defined in the beginning of the next Section. We next exploit the conservation of energy to see that the currents $j_{\widetilde{x}}$, at places $\widetilde{x}$ such that this condition is violated, in fact do not contribute to the evaluation of the conductivity. We finally take into account that $\epsilon \to 0$ before that $t \to \infty$ for Theorem 1, or that S generates a diffusion on time scales of order ϵ^n for Theorem 2, in order to get a bound on rest terms.

6

4 Proof of Theorems 1 and 2

The proofs of our two results are analogous but, since that of Theorem 2 is slightly more involved, we focus on it. The needed adaptations to show Theorem 1 are presented in the last Subsection.

Let us first define what are non-resonant frequencies near a bond $\widetilde{x} \in \widetilde{\mathbb{Z}}_d^N$. Let $\xi = (\xi_x)_{x \in \mathbb{Z}_N^d}$ with $\xi_x \in \mathbb{Z}$ for all $x \in \mathbb{Z}_N^d$. Given an integer $r \geq 0$, given $x \in \mathbb{Z}_N^d$ and given $\widetilde{x} = (a, b) \in \widetilde{\mathbb{Z}}_N^d$, we define

$$\mathrm{B}_1(x, r) \;=\; \{y \in \mathbb{Z}_N^d : |y - x|_1 \leq r\} \qquad \text{and} \qquad \mathrm{B}_1(\widetilde{x}, r) \;=\; \mathrm{B}_1(a, r) \cup \mathrm{B}_1(b, r).$$

Given $\widetilde{x} \in \widetilde{\mathbb{Z}}_N^d$, given a realization ω of the frequencies, and given an integer $r \geq 0$, we define

$$\langle \omega, \xi \rangle_{\widetilde{x}, r} \;=\; \sum_{y \in \mathrm{B}_1(\widetilde{x}, r)} \omega_y \xi_y.$$

For $\alpha, c \in \mathbb{R}$, for $\widetilde{x} \in \widetilde{\mathbb{Z}}_N^d$, and for an integer $r \geq 0$, we next define the diophantine set

$$D_{\alpha, c}(\widetilde{x}, r) \;=\; \left\{\omega : |\langle \omega, \xi \rangle_{\widetilde{x}, r}| \geq \frac{c}{|\xi|_1^\alpha} \text{ for all } \xi \neq 0 \text{ on } \mathrm{B}_1(\widetilde{x}, r)\right\}.$$

Here, $\xi \neq 0$ on $\mathrm{B}_1(\widetilde{x}, r)$ means that $\xi_x \neq 0$ for at least one $x \in \mathrm{B}_1(\widetilde{x}, r)$. The following monotony property holds: if $r \leq r'$, then $D_{\alpha, c}(\widetilde{x}, r) \subset D_{\alpha, c}(\widetilde{x}, r')$. It is known (see for example [25]) that, given $\delta > 0$ and an integer $r \geq 0$, there exist constants $\alpha < +\infty$ large enough and $c > 0$ small enough such that, for all $\widetilde{x} \in \widetilde{\mathbb{Z}}_N^d$, it holds that

$$\mathsf{P}_\star\big(\omega \in D_{\alpha, c}(\widetilde{x}, r)\big) \;\geq\; 1 - \delta. \tag{4}$$

From now, let $n \geq 1$ be as in the hypotheses. Let $\delta > 0$ to be fixed in Subsection 4.2, and let $\alpha, c > 0$ be such that (4) holds with $r = n$. We also fix a realization ω of the frequencies. All the constants introduced below may depend on n, δ, α and c.

4.1 Approximate solution to the Poisson equation

Let us start by a simple, but crucial, observation. We say that a function $u = u(q, p)$ on the phase space is p-symmetric if $u(q, p) = u(q, -p)$ for all $(q, p) \in (\mathbb{R}^2)^{N^d}$; we similarly define p-antisymmetric functions. The generators A_{har} and A_{anh} map p-symmetric functions to p-antisymmetric functions, and p-antisymmetric functions to p-symmetric functions. The interplay between p-symmetric and p-antisymmetric functions has shown to have deep consequences in an other context [4].

For the rest of this Subsection, we fix $\widetilde{x} \in \mathbb{Z}_N^d$, and we assume that $\omega \in D_{\alpha, c}(\widetilde{x}, n)$. For an integer $k \geq 0$, we designate by $||| \cdot |||_k$ the norm of the Sobolev space $\mathrm{H}^k(\mathbb{R}^{2N^d}, \langle \cdot \rangle_\beta)$; explicitly

$$|||u|||_k \;=\; \sum_{\alpha \in \mathbb{N}^{2N^d} : |\alpha|_1 \leq k} ||\partial^\alpha u||_{\mathrm{L}^2(\mathbb{R}^{2N^d}, \langle \cdot \rangle_\beta)} \quad \text{with} \quad |\alpha|_1 = \sum_{j=1}^{2N^d} \alpha_j \quad \text{and} \quad \partial^\alpha u = \partial_{x_1}^{\alpha_1} \ldots \partial_{x_{2N^d}}^{\alpha_{2N^d}}.$$

Let us first show the two following lemmas.

Lemma 1. *Let $\omega \in D_{\alpha, c}(\widetilde{x}, n)$. Let $f \in \mathcal{C}_{temp}^\infty(\mathbb{R}^{2N^d})$ be a p-antisymmetric function that depends only on the variables (q_y, p_y) with $y \in \mathrm{B}_1(\widetilde{x}, r)$ for some $0 \leq r \leq n - 1$. Then there exists a p-symmetric function*

7

$u \in \mathcal{C}^{\infty}_{temp}(\mathbb{R}^{2N^d})$, depending on the same variables as f, satisfying $\langle u \rangle_\beta = 0$, and solving

$$-A_{har} u \;=\; f.$$

Moreover, there exists a integer $m \geq 0$ such that, for any $k \geq 0$ and for some constant $C_k < +\infty$,

$$|||u|||_k \;\leq\; C_k \, |||f|||_{k+m}.$$

It follows from this Lemma that $A_{anh} u \in \mathcal{C}^{\infty}_{temp}(\mathbb{R}^{2N^d})$ is p-antisymmetric, and depends only on the variables (q_y, p_y) with $y \in B_1(\widetilde{x}, r+1)$. Moreover, there exists some non-negative integer m' such that, for any $k \geq 0$ and for some constant $C_k < +\infty$,

$$|||A_{anh} u|||_k \;\leq\; C_k \, |||f|||_{k+m'}.$$

Proof. For $z > 0$, the solution $u_z \in \mathcal{C}^{\infty}_{temp}(\mathbb{R}^{2N^d})$ to the equation

$$(z - A_{har}) u_z \;=\; f$$

exists and is unique ; it is given by

$$u_z \;=\; \int_0^\infty e^{-zt} e^{A_{har} t} f \, \mathrm{d}t \;=\; \int_0^\infty e^{-zt} f \circ X_t^{har} \, \mathrm{d}t \tag{5}$$

where $(X_t^{har})_{t \geq 0}$ is the deterministic process generated by A_{har}. Since $f \in \mathcal{C}^{\infty}_{temp}(\mathbb{R}^{2N^d})$, and since $(X_t^{har})_{t \geq 0}$ explicitly given by (7), it is checked that u_z defined by (5) is indeed an element of $\mathcal{C}^{\infty}_{temp}(\mathbb{R}^{2N^d})$. We will show that there exists a function u, having all the regularity properties appearing in the conclusions of the Lemma, depending on the same variables as f, and such that

$$\lim_{0 < z \to 0} u_z \;=\; u \qquad \text{and} \qquad \lim_{0 < z \to 0} A_{har} u_z \;=\; A_{har} u, \tag{6}$$

these limits being for example understood in $L^2(\mathbb{R}^{2N^d})$. This implies that u solves the equation $-A_{har} u = f$. So does then the function $u - \langle u \rangle_\beta$, which inherits also from the other properties of u mentioned in the conclusions of the Lemma. This will thus conclude the proof.

For given initial conditions $(q, p) \in \mathbb{R}^{2N^d}$, one has

$$X_t^{har}(q, p) \;=\; (\ldots, q_k \cos \omega_k t + (p_k/\omega_k) \sin \omega_k t, -\omega_k q_k \sin \omega_k t + p_k \cos \omega_k t, \ldots). \tag{7}$$

Therefore, defining

$$g(q, p; \theta) \;=\; f(\ldots, q_k \cos \theta_k + (p_k/\omega_k) \sin \theta_k, -\omega_k q_k \sin \theta_k + p_k \cos \theta_k, \ldots)$$

and

$$\theta(t) \;=\; \omega t \in \mathbb{R}^{N^d},$$

one has

$$u_z(q, p) \;=\; \int_0^\infty e^{-zt} g(q, p; \theta(t)) \, \mathrm{d}t.$$

8

It is convenient to write g in Fourier's variables:

$$g(q, p; \theta) = \sum_{\xi \in \mathbb{Z}^{Nd}} \hat{g}(q, p; \xi) \, e^{i\theta \cdot \xi} \quad \text{with} \quad \hat{g}(q, p; \xi) = \frac{1}{(2\pi)^{Nd}} \int_{[0, 2\pi]^{Nd}} g(q, p; \theta) \, e^{-i\theta \cdot \xi} \, d\theta.$$

Since f only depends on the variables (q_y, p_y) with $y \in B(\widetilde{x}, r)$, it holds that $\hat{g}(q, p, \xi) = 0$ for any (q, p), as soon as $\xi_y \neq 0$ for some $y \notin B(\widetilde{x}, r)$. Let us see that $\hat{g}(q, p; 0) = 0$ for any (q, p). For this, let us define $\overline{\theta} = \overline{\theta}(q, p) \in [0, 2\pi]^{Nd}$ by

$$q_k + ip_k/\omega_k = \left(q_k^2 + p_k^2/\omega_k^2\right)^{1/2} \exp(i\overline{\theta}_k).$$

The p-antisymmetry of f implies that

$$g(q, p; 2\overline{\theta} - \theta) = -g(q, p; \theta),$$

from which $\hat{g}(q, p; 0) = 0$ follows.

One now computes that

$$u_z(q, p) = \sum_{\xi \in \mathbb{Z}^{Nd} - \{0\}} \hat{g}(q, p; \xi) \int_0^\infty e^{-zt} e^{it\omega \cdot \xi} \, dt = \sum_{\xi \in \mathbb{Z}^{Nd} - \{0\}} \frac{\hat{g}(q, p; \xi)}{z - i\omega \cdot \xi}$$

and so, taking the limit $z \to 0$,

$$u(q, p) = i \sum_{\xi \in \mathbb{Z}^{Nd} - \{0\}} \frac{\hat{g}(q, p; \xi)}{\langle \omega \cdot \xi \rangle_{\widetilde{x}, r}}.$$

The regularity properties of u are obtained thanks to the hypothesis $\omega \in D_{\alpha, c}(\widetilde{x}, n)$, as is usual in KAM theory ; see for example [25]. From there, (6) follows. $\square$

Lemma 2. *Let* $f \in \mathcal{C}_{temp}^\infty(\mathbb{R}^{2N^d})$ *be a p-antisymmetric function that depends only on the variables* (q_y, p_y) *with* $y \in B_1(\widetilde{x}, r)$ *for some* $0 \leq r \leq n - 1$. *Then there exists a function* $u \in \mathcal{C}_{temp}^\infty(\mathbb{R}^{2N^d})$, *satisfying* $\langle u \rangle_\beta = 0$, *depending on the same variables as f, and solving*

$$-Su = f.$$

Moreover, for any $k \geq 0$ and for some constant $C_k < +\infty$, it holds that $|||u|||_k \leq C_k |||f|||_k$.

Proof. Let us fix the values of $|p_x|$ and q_x for all $x \in \mathbb{Z}_N^d$, so that in fact f is now seen as a function on $\{-1, +1\}^{|B_1(\widetilde{x}, r)|}$, the set of signs of the impulsions. The operator S generates a standard continuous time random walk on this set. Writing $S = |B_1(\widetilde{x}, r)|(T - Id)$, one has $u = \frac{1}{|B_1(\widetilde{x}, r)|} \sum_{k \geq 0} T^k f$. The terms $T^k f$ converge exponentially fast to 0, provided that f is of mean zero with respect to the uniform measure on $B_1(\widetilde{x}, r)$. So it is since f is p-antisymmetric. The regularity of u follows form this representation. $\square$

The function $j_{\widetilde{x}}$ is p-antisymmetric, belongs to $\mathcal{C}_{temp}^\infty(\mathbb{R}^{2N^d})$, and depends only on the variables in $B(\widetilde{x}, 0)$. Lemma 1 ensures that there exist functions $u_{\widetilde{x}}^{(1)}, \dots, u_{\widetilde{x}}^{(n)}$ solving the following finite hierarchy

$$-A_{har} u_{\widetilde{x}}^{(1)} = j_{\widetilde{x}}, \tag{8}$$

$$-A_{har} u_{\widetilde{x}}^{(2)} = A_{anh} u_{\widetilde{x}}^{(1)}, \tag{9}$$

$$\dots$$

$$-A_{har} u_{\widetilde{x}}^{(n)} = A_{anh} u_{\widetilde{x}}^{(n-1)}. \tag{10}$$

By Lemma 2, there exists also a function $v_{\widetilde{x}}^{(n)}$ such that

$$-S v_{\widetilde{x}}^{(n)} \;=\; A_{anh} u_{\widetilde{x}}^{(n)}.$$

Defining

$$u_{\widetilde{x}} \;=\; u_{\widetilde{x}}^{(1)} + \epsilon u_{\widetilde{x}}^{(2)} + \cdots + \epsilon^{n-1} u_{\widetilde{x}}^{(n)} \qquad \text{and} \qquad v_{\widetilde{x}} \;=\; u_{\widetilde{x}} - v_{\widetilde{x}}^{(n)}$$

one finds that

$$j_{\widetilde{x}} \;=\; -L u_{\widetilde{x}} + \epsilon^n (A_{anh} u_{\widetilde{x}}^{(n)} + S u_{\widetilde{x}}) \;=\; -L u_{\widetilde{x}} + \epsilon^n S v_{\widetilde{x}}. \tag{11}$$

The functions $u_{\widetilde{x}}$ and $v_{\widetilde{x}}$ belong to $\mathcal{C}^{\infty}(\mathbb{R}^{2N^d})$, are of mean zero, depend only on the variables (q_y, p_y) with $y \in \mathrm{B}_1(\widetilde{x}, n)$, and have (Sobolev) norms that do not depend on $\widetilde{x}$, as long as $\omega \in D_{\alpha,c}(\widetilde{x}, n)$.

Remark. While it is not difficult to solve the finite hierarchy (8)-(9) as we did under a non-resonant assumption on ω, one could in fact have expected it to be much less straightforward. Indeed, the dynamics generated by A_{har} is very degenerated since it preserves the energy of every single particle. This has two consequences. First, the equation $-A_{har} u = f$ can only be solved if f is of mean zero with respect to every microcanonical surface defined by the value of the energies of each uncoupled particle. Second, if this is the case, then u is not unique, since every function depending only on the energies of decoupled particles lies in the kernel of A_{har}. The fact that the currents are p-antisymmetric, and that A_{anh} interchanges p-symmetric and p-antisymmetric functions, guarantees therefore that the zero-mean condition is satisfied at all orders, without that we have to select a particular solution at each step. The operator S instead destroys this symmetry of the system, allowing diffusion of energy.

4.2 Resonances

We fix a realization ω of the frequencies. We let

$$A_N \;=\; \{ \widetilde{x} \in \widetilde{\mathbb{Z}}_N^d : \omega \in D_{\alpha,c}(\widetilde{x}, n) \}.$$

We then partition $\widetilde{\mathbb{Z}}_N^d - A_N$ as

$$\widetilde{\mathbb{Z}}_N^d - A_N \;=\; \bigcup_{\gamma} \mathcal{C}_{\gamma},$$

where $(\mathcal{C}_{\gamma})_{\gamma}$ is the set of connected components, or clusters, of $\widetilde{\mathbb{Z}}_N^d - A_N$. We let $\ell(\mathcal{C}_{\gamma}) \geq 1$ be the length of the longest self-avoiding path in $\mathcal{C}_{\gamma}$. Given $\widetilde{x} \in \widetilde{\mathbb{Z}}_N^d - A_N$, we also define $\mathcal{C}(\widetilde{x})$ as the cluster $\mathcal{C}_{\gamma}$ such that $\widetilde{x} \in \mathcal{C}_{\gamma}$.

In the sequel, by a slight abuse of notation, we will say that $x \in \mathbb{Z}_N^d$ belongs to $E \subset \widetilde{\mathbb{Z}}_N^d$, if there exists a bond $\widetilde{x} \in E$ such that x is a component of $\widetilde{x}$. So any subset $E \subset \widetilde{\mathbb{Z}}_N^d$ will also be considered as a subset of $\mathbb{Z}_N^d$. Given $\mathcal{K} \subset \mathbb{Z}_N^d$, we define its border $\partial \mathcal{K} \subset \widetilde{\mathbb{Z}}_N^d$ as as the set of bonds $\widetilde{x} = (a, b)$ such that

$$\partial \mathcal{K} \;=\; \{ \widetilde{x} = (a, b) \in \widetilde{\mathbb{Z}}_N^d : (a \in \mathcal{K} \text{ and } b \notin \mathcal{K}) \quad \text{or} \quad (a \notin \mathcal{K} \text{ and } b \in \mathcal{K}) \}.$$

Given $\mathcal{K} \subset \widetilde{\mathbb{Z}}_N^d$, one defines $\partial \mathcal{K}$ as the border of $\mathcal{K}$ seen as a subset of $\widetilde{\mathbb{Z}}_N^d$. The following Lemma contains the core of the argument that allows us to get rid of resonances.

10

Lemma 3. *Let $\mathcal{C}_\gamma \subset \widetilde{\mathbb{Z}}_N^d - A_N$ be a cluster, and let $\ell(\mathcal{C}_\gamma) = m$. It holds that*

$$\sum_{\widetilde{x} \in \mathcal{C}_\gamma} j_{\widetilde{x}} \;=\; \sum_{\widetilde{x} \in \partial \mathcal{C}_\gamma} \theta_1(\widetilde{x}, \gamma) \, j_{\widetilde{x}} - \epsilon^{-1} L \Big(\sum_{x \in \mathcal{C}_\gamma \subset \mathbb{Z}_N^d} \theta_2(x) \, e_x \Big)$$

where θ_1 and θ_2 are integer valued functions that satisfy the bounds

$$\sup_{\widetilde{x} \in \partial \mathcal{C}_\gamma} |\theta_1(\widetilde{x}, \gamma)| \;\leq\; \mathrm{C} \, m^d, \qquad \sup_{x \in \mathcal{C}_\gamma} |\theta_2(x)| \;\leq\; \mathrm{C} \, m^d,$$

for a constant C that depends only on the dimension d.

Proof. Given $\mathcal{K} \subset \mathbb{Z}_N^d$, the conservation of energy implies

$$\epsilon^{-1} L \Big(\sum_{x \in \mathcal{K}} e_x \Big) \;=\; \sum_{\widetilde{x} \in \partial \mathcal{K}} \zeta(\widetilde{x}) \, j_{\widetilde{x}}. \tag{12}$$

Here, writing $\widetilde{x} = (a, b)$, one has $\zeta(\widetilde{x}) = 1$ if $b \in \mathcal{K}$ and $\zeta(\widetilde{x}) = -1$ if $a \in \mathcal{K}$. To apply this formula, we can repeatedly consider contours that partition $\mathcal{C}_\gamma$ into two pieces, and get rid of all the currents in $\mathcal{C}_\gamma$. A systematical way to do this is as follows (see Figure 1). Let us give a direction $k \in \{1, \dots, d\}$ and a value $s \in \mathbb{Z}_N$ such that for some $\widetilde{x} = (a, a + \mathsf{e}_k) \in \mathcal{C}_\gamma$, it holds that $a_k = s$, where a_k denotes the kth coordinate of a. We then define

$$P(s, k) \;=\; \{\widetilde{x} = (a, a + \mathsf{e}_k) \in \mathcal{C}_\gamma : a_k = s\},$$

as well as the set $\mathcal{K}(s, k) \subset \mathcal{C}_\gamma \subset \mathbb{Z}_N^d$ made of points x such that $x_k > s$. Then, using (12),

$$\epsilon^{-1} L \Big(\sum_{x \in \mathcal{K}(s,k)} e_x \Big) \;=\; \sum_{\widetilde{x} \in P(s,k)} j_{\widetilde{x}} + \sum_{\widetilde{x} \in \partial \mathcal{K}(s,k) - P(s,k)} \zeta(\widetilde{x}) \, j_{\widetilde{x}}.$$

It hols that $\partial \mathcal{K}(s, k) - P(s, k) \subset \partial \mathcal{C}_\gamma$. This operation can be repeated with every direction $k \in \{1, \dots, d\}$ and every value $s \in \mathbb{Z}_N$. This only needs to be done $\mathcal{O}(m^d)$ times to cover the whole cluster ; summing one obtains

$$-\epsilon^{-1} L \Big(\sum_{x \in \mathcal{C}_\gamma} \theta_2(x) e_x \Big) \;=\; \sum_{\widetilde{x} \in \mathcal{C}_\gamma} j_{\widetilde{x}} - \sum_{\widetilde{x} \in \partial \mathcal{C}_\gamma} \theta_1(\widetilde{x}, \gamma) \, j_{\widetilde{x}},$$

for some functions θ_1 and θ_2 that satisfy the stated bounds. $\square$

Thanks to this Lemma, and using the fact that $L \langle e \rangle_\beta = 0$, we decompose the total current as follows

$$J_N \;=\; \sum_{\widetilde{x} \in A_N} j_{\widetilde{x}} + \sum_\gamma \Big(\sum_{\widetilde{x} \in \mathcal{C}_\gamma} j_{\widetilde{x}} \Big) = \sum_{\widetilde{x} \in A_N} j_{\widetilde{x}} + \sum_\gamma \Big(\sum_{\widetilde{x} \in \partial \mathcal{C}_\gamma} \theta_1(\widetilde{x}) j_{\widetilde{x}} \Big) - \epsilon^{-1} L \Big(\sum_\gamma \Big(\sum_{x \in \mathcal{C}_\gamma} \theta_2(x) \, e_x \Big) \Big)$$

$$=: \sum_{\widetilde{x} \in A_N} \psi_1(\widetilde{x}) \, j_{\widetilde{x}} - \epsilon^{-1} L \Big(\sum_{x \in \mathbb{Z}_N^d} \psi_2(x) \, (e_x - \langle e \rangle_\beta) \Big). \tag{13}$$

Here

$$\psi_1(\widetilde{x}) \;=\; 1 + \sum_{\gamma : \widetilde{x} \in \partial \mathcal{C}_\gamma} \theta_1(\widetilde{x}, \gamma),$$

the sum containing two non-zero terms at most, and

$$\psi_2(x) \;=\; \theta_2(x) \text{ if } x \in \bigcup_\gamma \mathcal{C}_\gamma \qquad \text{and} \qquad \psi_2(x) \;=\; 0 \text{ otherwise.}$$
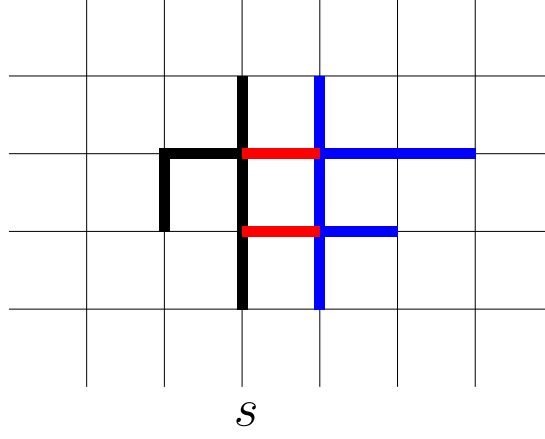
Figure 1: The sets $\mathcal{C}_\gamma$, $P(s, k)$ and $\mathcal{K}(s, k)$ appearing in the proof of Lemma 3. Here $d = 2$ and $k = 1$. The whole figure represent a cluster $\mathcal{C}_\gamma$. For a given s, the bonds in red constitute the set $P(s, k = 1)$, while the bonds in blue constitute the set $\mathcal{K}(s, k = 1)$.

The functions ψ_1 and ψ_2 inherit the properties of θ_1 and θ_2 stated in Lemma 3. If $\widetilde{x} \notin \bigcup_\gamma \partial \mathcal{C}_\gamma$, then $|\psi_1(\widetilde{x})| = 1$, whereas if $\widetilde{x} \in \bigcup_\gamma \partial \mathcal{C}_\gamma$, then $|\psi_1(\widetilde{x})| \leq \mathrm{C}\, m^d$, where $m = \ell(\mathcal{C}_\gamma)$, where $\mathcal{C}_\gamma$ is the largest cluster such that $\widetilde{x} \in \partial \mathcal{C}_\gamma$. Similarly, if $x \notin \bigcup_\gamma \mathcal{C}_\gamma$, then $|\psi_2(x)| = 0$, while if $x \in \mathcal{C}_\gamma$ for some γ, then $|\psi_2(x)| \leq \mathrm{C}\, m^d$, with $m = \ell(\mathcal{C}_\gamma)$.

We now may fix the parameter δ introduced at the beginning of this Section. We take $\delta > 0$ small enough so that there exist constants $\mathrm{C} < +\infty$ and $c' > 0$ such that, for every integer $m \geq 1$, and for every $\widetilde{x} \in \mathbb{Z}_N^d$, it holds

$$\mathsf{P}_\star\big(\ell(\mathcal{C}(\widetilde{x})) = m\big) \ \leq \ \mathrm{C}\, e^{-c' m}. \tag{14}$$

The number δ is well defined. To see this, we first notice that, for $\widetilde{x}, \widetilde{y} \in \widetilde{\mathbb{Z}}_N^d$, the events $\omega \notin D_{\alpha,c}(\widetilde{x}, n)$ and $\omega \notin D_{\alpha,c}(\widetilde{y}, n)$ become independent as soon as $\mathrm{B}_1(\widetilde{x}, n) \cap \mathrm{B}_1(\widetilde{y}, n) = 0$. Next, we observe that the number of self-avoiding paths in $\widetilde{\mathbb{Z}}_N^d$ of length m is bounded by $(2d - 1)^m$. But the number of disjoints balls of fixed radius n in $\widetilde{\mathbb{Z}}_N^d$, with center contained in any such path, is proportional to m as $m \to \infty$. Therefore, the probability of forming a path of length m in $\widetilde{\mathbb{Z}}_N^d - A_N$ decays exponentially with m if δ is small enough.

Lemma 4. *For every $\widetilde{x} \in \widetilde{\mathbb{Z}}_N^d$, we let $f_{\widetilde{x}}$ be a smooth function on the phase space $\mathbb{R}^{2N^d}$, that satisfies $\langle f_{\widetilde{x}} \rangle_\beta = 0$, that depends only on the variables in $\mathrm{B}(\widetilde{x}, n)$, and which H^1-norm can be bounded by a constant independent of $\widetilde{x}$. Then, there exists a constant $\mathrm{C} < +\infty$, such that, almost surely,*

$$\limsup_{N \to \infty} \left\langle \left(\frac{1}{N^{d/2}} \sum_{\widetilde{x} \in A_N} \psi_1(\widetilde{x})\, f_{\widetilde{x}} \right)^2 \right\rangle_\beta \ \leq \ \mathrm{C}, \tag{15}$$

$$\limsup_{N \to \infty} \left\langle \left(\frac{1}{N^{d/2}} \sum_{x \in \mathbb{Z}_N^d} \psi_2(x)\, \big(e_x - \langle e \rangle_\beta\big) \right)^2 \right\rangle_\beta \ \leq \ \mathrm{C}. \tag{16}$$

Proof. Throughout the proof of this lemma, $\mathrm{C} < +\infty$ and $c' > 0$ denote constants. The proof of both bounds are analogous, and we only handle the first one. Expanding the square and applying Cauchy-

Schwarz inequality yields

$$V_N := \left\langle \left(\frac{1}{N^{d/2}} \sum_{\widetilde{x} \in A_N} \psi_1(\widetilde{x}) \, f_{\widetilde{x}} \right)^2 \right\rangle_\beta \leq \frac{1}{N^d} \sum_{\widetilde{x}, \widetilde{y} \in A_N} \psi_1(\widetilde{x}) \psi_1(\widetilde{y}) \, |\langle f_{\widetilde{x}} f_{\widetilde{y}} \rangle_\beta| \leq \frac{1}{N^d} \sum_{\widetilde{x}, \widetilde{y} \in A_N} \psi_1^2(\widetilde{x}) \, |\langle f_{\widetilde{x}} f_{\widetilde{y}} \rangle_\beta|$$

Then, first, thanks to the bounds on $|\psi_1(\widetilde{x})|$ stated after the proof of Lemma 3, and thanks to the exponential bound (14) on the probability of forming clusters of size m, it holds that

$$\mathsf{E}_\star\big(|\psi_1(\widetilde{x})|^p\big) \leq \mathrm{C}_p, \tag{17}$$

for any $p \geq 1$, and for some constant $\mathrm{C}_p < +\infty$ independent of $\widetilde{x} \in \widetilde{\mathbb{Z}}_N^d$. Second, thanks to the decorrelation bound (1) on the Gibbs measure, and to the hypotheses on the functions $f_{\widetilde{x}}$, it holds that

$$\rho_{\widetilde{x}} := \sum_{\widetilde{y} \in A_N} |\langle f_{\widetilde{x}} f_{\widetilde{y}} \rangle_\beta| \leq \mathrm{C}.$$

Therefore

$$V_N \leq \frac{1}{N^d} \sum_{\widetilde{x} \in A_N} \big(\psi_1^2(\widetilde{x}) - \mathsf{E}_\star(\psi_1^2(\widetilde{x})) \big) \rho_{\widetilde{x}} + \frac{1}{N^d} \sum_{\widetilde{x} \in A_N} \mathsf{E}_\star(\psi_1^2(\widetilde{x})) \, \rho_{\widetilde{x}} =: Y_N + Z_N.$$

Since Z_N is a bounded deterministic sequence, it suffices now to bound Y_N. By the Borel-Cantelli lemma and Markov inequality, the proof will be concluded if one finds $p \geq 1$ such that

$$\sum_{N \geq 1} \mathsf{E}_\star\big(|Y_N|^p\big) < +\infty. \tag{18}$$

If $d = 1$, one can take $p = 4$, but if $d \geq 2$, the choice $p = 2$ suffices. We only will deal with this second case ; the case $d = 1$ is obtained similarly at the price of slightly longer computations. To simplify some further notations, let us write

$$\phi(\widetilde{x}) = \big(\psi_1^2(\widetilde{x}) - \mathsf{E}_\star(\psi_1^2(\widetilde{x})) \big).$$

One has

$$\mathsf{E}_\star\big(Y_N^2\big) \leq \frac{\mathrm{C}}{N^{2d}} \sum_{\widetilde{x}, \widetilde{y} \in A_N} \mathsf{E}_\star\big(\phi(\widetilde{x}) \phi(\widetilde{y})\big). \tag{19}$$

For $\widetilde{x}, \widetilde{y} \in \widetilde{\mathbb{Z}}_N^d$, let us set $|\widetilde{x} - \widetilde{y}|_1 = \min\{|x - y|_1 : x \in \widetilde{x}, y \in \widetilde{y}\}$. By (17), the terms in the sum in the right hand side of (19) are uniformly bounded, and it now suffices to establish decay bounds on off-diagonal terms. Let us fix some $\widetilde{x}, \widetilde{y} \in A_N$. The claims below are valid for $|\widetilde{x} - \widetilde{y}|_1$ large enough. Given $c'' > 0$, we then define an event E as follows: $\omega \in E$ if nor $\widetilde{x}$ nor $\widetilde{y}$ do belong to the border of a cluster of size larger or equal to $c''|\widetilde{x} - \widetilde{y}|$. Then, if c'' small enough, the variables $\phi(\widetilde{x})$ and $\phi(\widetilde{y})$ become independent with respect to $\mathsf{P}_\star(\cdot|E)$. Therefore

$$\begin{aligned}
\mathsf{E}_\star\big(\phi(\widetilde{x})\phi(\widetilde{y})\big) &= \mathsf{E}_\star\big(\phi(\widetilde{x})\phi(\widetilde{y})\big|E\big)\mathsf{P}_\star(E) + \mathsf{E}_\star\big(\phi(\widetilde{x})\phi(\widetilde{y})\big|E^c\big)\mathsf{P}_\star(E^c) \\
&= \mathsf{E}_\star\big(\phi(\widetilde{x})\big|E\big)\mathsf{E}_\star\big(\phi(\widetilde{y})\big|E\big)\mathsf{P}_\star(E) + \mathsf{E}_\star\big(\phi(\widetilde{x})\phi(\widetilde{y})\big|E^c\big)\mathsf{P}_\star(E^c) \\
&= -\mathsf{E}_\star\big(\phi(\widetilde{x})\big|E\big)\mathsf{E}_\star\big(\phi(\widetilde{y})\big|E^c\big)\mathsf{P}_\star(E^c) + \mathsf{E}_\star\big(\phi(\widetilde{x})\phi(\widetilde{y})\big|E^c\big)\mathsf{P}_\star(E^c),
\end{aligned}$$

13

where one has used the fact that $\mathsf{E}_\star(\phi(\widetilde{y})) = 0$ to get the last line. The exponential bound $\mathsf{P}_\star(E^c) \leq \mathrm{C}\,e^{-c'|\widetilde{x}-\widetilde{y}|_1}$ holds, so that in particular $\mathsf{P}_\star(E)$ is bounded form below by a strictly positive constant. So it holds that

$$|\mathsf{E}_\star(\phi(\widetilde{x})|E)| \leq \mathrm{C}|\mathsf{E}_\star(\phi(\widetilde{x}))| \leq \mathrm{C},$$

$$|\mathsf{E}_\star(\phi(\widetilde{y})|E^c)\mathsf{P}_\star(E^c)| \leq \mathsf{E}_\star(\phi^2(\widetilde{y}))^{1/2}\mathsf{P}_\star^{1/2}(E^c) \leq \mathrm{C}\,e^{-c'|\widetilde{x}-\widetilde{y}|_1},$$

$$|\mathsf{E}_\star(\phi(\widetilde{x})\phi(\widetilde{y})|E^c)\mathsf{P}_\star(E^c)| \leq \mathsf{E}_\star(\phi^2(\widetilde{x})\phi^2(\widetilde{y}))^{1/2}\mathsf{P}_\star^{1/2}(E^c) \leq \mathrm{C}\,e^{-c'|\widetilde{x}-\widetilde{y}|_1}.$$

Thus

$$|\mathsf{E}_\star(\phi(\widetilde{x})\phi(\widetilde{y}))| \leq \mathrm{C}\,e^{-c'|\widetilde{x}-\widetilde{y}|_1}.$$

Inserting this estimate in (19), it follows that $\mathsf{E}_\star(Y_N^2) \leq \mathrm{C}/N^d$. Since we have assumed $d \geq 2$, this in turn yields (18), and so concludes the proof. $\square$

4.3 Concluding the proof of Theorem 2

We fix a realization ω of frequencies. We combine (11) and (13) to get

$$J_N = -L\left(\sum_{\widetilde{x}\in A_N}\psi_1(\widetilde{x})\,u_{\widetilde{x}}\right) - \epsilon^{-1}L\left(\sum_{\widetilde{x}\in\mathbb{Z}_N^d}\psi_2(x)\,(e_x - \langle e\rangle_\beta)\right) + \epsilon^n S\left(\sum_{\widetilde{x}\in A_N}\psi_1(\widetilde{x})\,v_{\widetilde{x}}\right).$$

Integrating over time and dividing by $t^{1/2}N^{d/2}$, one finds two martingales M_t and N_t such that

$$\frac{1}{\sqrt{t}}\int_0^t \mathcal{J}_N \circ X_s\,\mathrm{d}s = M_t + \frac{1}{t^{1/2}N^{d/2}}\sum_{\widetilde{x}\in A_N}\psi_1(\widetilde{x})\,(u_{\widetilde{x}} - u_{\widetilde{x}}\circ X_t)$$

$$+ N_t + \frac{\epsilon^{-1}}{t^{1/2}N^{d/2}}\sum_{x\in\mathbb{Z}_N^d}\psi_2(x)\left((e_x - \langle e\rangle_\beta)\circ X_t - (e_x - \langle e\rangle_\beta)\right)$$

$$+ \frac{\epsilon^n}{\sqrt{t}}\int_0^t S\left(\frac{1}{N^{d/2}}\sum_{\widetilde{x}\in A_N}\psi_1(\widetilde{x})\,v_{\widetilde{x}}\circ X_s\right)\mathrm{d}s.$$

We now separately estimate the variance at equilibrium of the five terms in the right hand side of this equation. We remind that, by the result of Subsection 4.1, the zero-mean functions $u_{\widetilde{x}}$ and $v_{\widetilde{x}}$ are smooth and local, uniformly in $\widetilde{x} \in A_N$. All the estimates below hold almost surely in the limit $N \to \infty$.

Let us start with the two martingales. First, by (15),

$$\mathsf{E}_\beta(M_t^2) = \left\langle\left(\frac{1}{N^{d/2}}\sum_{\widetilde{x}\in A_N}\psi_1(\widetilde{x})u_{\widetilde{x}}\right)\left(\frac{-\epsilon^n S}{N^{d/2}}\sum_{\widetilde{x}\in A_N}\psi_1(\widetilde{x})u_{\widetilde{x}}\right)\right\rangle_\beta$$

$$\leq \epsilon^n\left\langle\left(\frac{1}{N^{d/2}}\sum_{\widetilde{x}\in A_N}\psi_1(\widetilde{x})u_{\widetilde{x}}\right)^2\right\rangle_\beta^{1/2}\left\langle\left(\frac{-S}{N^{d/2}}\sum_{\widetilde{x}\in A_N}\psi_1(\widetilde{x})u_{\widetilde{x}}\right)^2\right\rangle_\beta^{1/2} = \mathcal{O}(\epsilon^n).$$

Next, since $Se_x = 0$ for every $x \in \mathbb{Z}_N^d$,

$$\mathsf{E}_\beta(N_t^2) = \left\langle\left(\frac{\epsilon^{-1}}{N^{d/2}}\sum_{x\in\mathbb{Z}_N^d}\psi_2(x)(e_x - \langle e\rangle_\beta)\right)\left(\frac{-\epsilon^{n-1}S}{N^{d/2}}\sum_{x\in\mathbb{Z}_N^d}\psi_2(x)(e_x - \langle e\rangle_\beta)\right)\right\rangle_\beta = 0.$$

Let us then consider the two terms not involving the operator S. By (15) and (16) and the stationarity of the measure $\langle\cdot\rangle_\beta$,

$$\mathsf{E}_\beta\left(\frac{1}{t^{1/2}N^{d/2}}\sum_{\widetilde{x}\in A_N}\psi_1(\widetilde{x})\,(u_{\widetilde{x}} - u_{\widetilde{x}}\circ X_t)\right)^2 \leq \frac{2}{t}\left\langle\left(\frac{1}{N^{d/2}}\sum_{\widetilde{x}\in A_N}\psi_1(\widetilde{x})u_{\widetilde{x}}\right)^2\right\rangle_\beta = \mathcal{O}(t^{-1})$$

and similarly

$$\mathsf{E}_\beta\left(\frac{\epsilon^{-1}}{t^{1/2}N^{d/2}}\sum_{x\in\mathbb{Z}_N^d}\psi_2(x)\left(e_x-e_x\circ X_t\right)\right)^2 \;=\; \mathcal{O}(\epsilon^{-2}t^{-1}).$$

One finally deals with the last term. By a classical bound [17], there exists a universal constant $C < +\infty$ such that

$$\mathsf{E}_\beta\left(\frac{\epsilon^n}{\sqrt{t}}\int_0^t S\left(\frac{1}{N^{d/2}}\sum_{\widetilde{x}\in A_N}\psi_1(\widetilde{x})\,v_{\widetilde{x}}\circ X_s\right)\mathrm{d}s\right)^2$$

$$\leq\; C\epsilon^{2n}\left\langle\left(\frac{S}{N^{d/2}}\sum_{\widetilde{x}\in A_N}\psi_1(\widetilde{x})\,v_{\widetilde{x}}\right)(-\epsilon^n S)^{-1}\left(\frac{S}{N^{d/2}}\sum_{\widetilde{x}\in A_N}\psi_1(\widetilde{x})\,v_{\widetilde{x}}\right)\right\rangle_\beta \;=\; \mathcal{O}(\epsilon^n),$$

where (15) has been used to get the last bound. The proof is concluded by taking the limit $t\to\infty$. $\square$

4.4   Proving Theorem 1

Since the proof of Theorem 1 is much similar to that of Theorem 2, we only indicate the main steps. We let $n\geq 1$. It is enough to consider the case $m=n$; we then need to take the limit of the following quantity:

$$\beta^2\left\langle\left(\frac{\epsilon}{\sqrt{t}}\int_0^{\epsilon^{-n}t}\mathcal{J}_N\circ X_s^{(\epsilon,0)}\,\mathrm{d}s\right)^2\right\rangle_\beta.$$

As in the proof of Theorem 1, we let $\delta > 0$ as fixed in Subsection 4.2, and we let $\alpha, c > 0$ be such that (4) holds with $r = n$. We also fix a realization ω of the frequencies. Assuming that $\omega\in D_{\alpha,c}(\widetilde{x},n)$, we write as in Subsection 4.1, that

$$j_{\widetilde{x}} \;=\; -Lu_{\widetilde{x}}+\epsilon^n A_{anh}u_{\widetilde{x}}^{(n)} \;=:\; -Lu_{\widetilde{x}}+\epsilon^n w_{\widetilde{x}}.$$

All the results of Subsection 4.2 still hold, and we combine this with the decomposition (13) of the total current, to get

$$\frac{\epsilon}{\sqrt{t}}\int_0^{\epsilon^{-n}t}\mathcal{J}_N\,\mathrm{d}s \;=\; \frac{\epsilon}{\sqrt{t}}\int_0^{\epsilon^{-n}t}\frac{-L}{N^{d/2}}\sum_{\widetilde{x}\in A_N}\psi_1(\widetilde{x})u_{\widetilde{x}}\,\mathrm{d}s + \frac{1}{\sqrt{t}}\int_0^{\epsilon^{-n}t}\frac{-L}{N^{d/2}}\sum_{x\in\mathbb{Z}_N^d}\psi_2(x)\left(e_x-\langle e\rangle_\beta\right)\mathrm{d}s$$

$$+\frac{\epsilon^{n+1}}{\sqrt{t}}\int_0^{\epsilon^{-n}t}\frac{1}{N^{d/2}}\sum_{\widetilde{x}\in A_N}\psi_1(\widetilde{x})w_{\widetilde{x}}\,\mathrm{d}s.$$

We then obtain bounds valid almost surely in the limit $N\to\infty$. Since the generator L is antisymmetric, the variances of the terms involving the generator L read

$$\left\langle\left(\frac{\epsilon}{\sqrt{t}}\int_0^{\epsilon^{-n}t}\frac{-L}{N^{d/2}}\sum_{\widetilde{x}\in A_N}\psi_1(\widetilde{x})u_{\widetilde{x}}\right)^2\right\rangle_\beta \;=\; \left\langle\left(\frac{-\epsilon}{\sqrt{t}}\frac{1}{N^{d/2}}\sum_{\widetilde{x}\in A_N}\psi_1(\widetilde{x})\left(u_{\widetilde{x}}\circ X_{\epsilon^{-n}t}-u_{\widetilde{x}}\right)\right)^2\right\rangle_\beta$$

$$=\; \mathcal{O}(\epsilon^2 t^{-1}),$$

$$\left\langle\left(\frac{1}{\sqrt{t}}\int_0^{\epsilon^{-n}t}\frac{L}{N^{d/2}}\sum_{x\notin A_N}\psi_2(x)e_x\right)^2\right\rangle_\beta \;=\; \mathcal{O}(t^{-1}).$$

Finally, by Jensen's inequality

$$\left\langle \left(\frac{\epsilon^{n+1}}{\sqrt{t}} \int_0^{\epsilon^{-n}t} \frac{1}{N^{d/2}} \sum_{\widetilde{x} \in A_N} \psi_1(\widetilde{x}) w_{\widetilde{x}} \, ds \right)^2 \right\rangle_\beta \leq \epsilon t \left\langle \left(\frac{1}{\epsilon^{-n}t} \int_0^{\epsilon^{-n}t} \left(\frac{1}{N^{d/2}} \sum_{\widetilde{x} \in A_N} \psi_1(\widetilde{x}) w_{\widetilde{x}} \right)^2 ds \right) \right\rangle_\beta$$

$$= \epsilon t \left\langle \left(\frac{1}{N^{d/2}} \sum_{\widetilde{x} \in A_N} \psi_1(\widetilde{x}) w_{\widetilde{x}} \right)^2 \right\rangle_\beta = \mathcal{O}(\epsilon t).$$

The proof is achieved by performing successively the limits $\epsilon \to 0$ and then $t \to \infty$. $\square$

5 Related models

We show how to extend our result to two different physical systems, and suggest an analogy with a chain of anharmonic oscillators. We only deal with the weak coupling limit, and clearly indicate where some pieces of information are missing to get true mathematical statements.

5.1 Weakly perturbed one-dimensional disordered harmonic chain

In [12], a one-dimensional disordered harmonic chain perturbed by small anharmonic interactions is considered. The phase space is the set of points $(q, p) \in \mathbb{R}^{2N}$, and the hamiltonian is of the form

$$H(q, p) = \frac{1}{2} \sum_{x \in \mathbb{Z}_N} \left(p_x^2 + \omega_x^2 q_x^2 \right) + \frac{1}{2} \sum_{x \in \mathbb{Z}_N} (q_x - q_{x+1})^2 + \frac{\epsilon}{4} \sum_{x \in \mathbb{Z}_N} q_x^4 + \frac{\epsilon}{4} \sum_{x \in \mathbb{Z}_N} (q_x - q_{x+1})^4 \qquad (20)$$

$$= H_{har} + \epsilon H_{anh}.$$

The hypotheses on $(\omega_x)_x$ are as in Section 2.[2] The time evolution is predicted by the usual hamiltonian equations. At $\epsilon = 0$, the chain is harmonic, and any state can be written as a linear combinations of the eigenmodes $(\xi^k)_{1 \leq k \leq N}$ of the systems ; moreover, due to everywhere onsite pinning and since the chain is one-dimensional, all these modes are typically exponentially localized. See for example [6] and references therein. Let us also denote by $(\overline{\omega}_k)_{1 \leq k \leq N}$ the eigenfrequencies of the modes, which, as opposed to $(\omega_x)_x$, do not form a sequence of independent and identically distributed variables. At least in a naive picture of Anderson localization, it should be however reasonable to think that the frequency $\overline{\omega}_1$ of a mode ξ_1 localized near a point $x_1 \in \mathbb{Z}_N$, and a frequency $\overline{\omega}_2$ of a mode ξ_2 localized near a point $x_2 \in \mathbb{Z}_N$, quickly decorrelate as $|x_1 - x_2|$ grows. This should be needed to get rid of resonances as we did in Subsection 4.2.

In this model as well, the generators A_{har} and A_{anh} exchange p-symmetric and p-antisymmetric functions. Now, the crux of the matter is that, for a function f that is p-antisymmetric, and that only depends on variables around some site x, the Poisson equation

$$-A_{har} u = f$$

should be solved in the same way as we did, with a solution u that is exponentially localized near x. Let us mention that this equation has been solved in [6] in the case where f is the current corresponding

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[2] In [12], randomness is on the masses and not on the pinning. This however should not make any crucial difference.

to a harmonic interaction. This case is quite special however, since there, a lucky cancellation makes disappear all possible resonances.

Looking at the proof of Lemma 1, one has indeed

$$u(q,p) \;=\; \lim_{0 < z \to 0} \int_0^\infty \mathrm{e}^{-zt} f \circ X_t^{har}(q,p)\, \mathrm{d}t.$$

This can be made more explicit since, using (4.4) and (4.5) in [6], one can write

$$f \circ X_t^{har}(q,p) \;=\; f\Big(\ldots, \sum_{j=1}^N \Big(\langle q, \xi^j \rangle \cos \overline{\omega}_j t + (\langle p, \xi^j \rangle / \overline{\omega}_j) \sin \overline{\omega}_j t \Big), $$
$$\sum_{j=1}^N \Big(-\overline{\omega}_j \langle q, \xi^j \rangle \cos \overline{\omega}_j t + \langle p, \xi^j \rangle \sin \overline{\omega}_j t \Big), \ldots \Big).$$

Here as well, one can define $g(q,p;\theta)$ by replacing $\overline{\omega}t$ by θ in this last formula. Defining then an angle $\overline{\theta}$ such that $\overline{\theta}_j$ is the phase of $\langle q, \xi^j \rangle + i \langle p, \xi^j \rangle / \overline{\omega}_j$, one then finds that the p-antisymmetry of f translates into the relation $g(q,p,2\overline{\theta} - \theta) = -g(q,p,\theta)$, which ensures that the Fourier expansion has no constant mode. Finally, the exponential localization of the eigenmodes should guarantee that only a small number of eigenfrequencies are practically involved, so that diophantine estimates can be used. Moreover, while the solution u itself will not be local anymore, it will be exponentially localized, and the limit $\epsilon \to 0$ should allow us to get rid of exponential tails.

Remark. For the chain defined by (20), it is known [2] that the Green-Kubo conductivity $\kappa(\beta, \epsilon)$, seen as function of the inverse temperature β and the coupling strength ϵ, satisfies the exact scaling

$$\kappa(\beta, \epsilon) \;=\; \kappa(\beta/r, \epsilon/r) \quad \text{for any} \quad r > 0.$$

Therefore, the behavior of the conductivity in the small coupling regime corresponds to its behavior at low temperatures. Many numerical works deal with the diffusion of an initially localized energy packet (diffusion at "zero temperature"). The heat equation can sometimes serve as a good phenomenological model to describe the spreading of this packet [22]. If one assumes that the diffusion constant in this equation is given by the Green-Kubo conductivity[3], then our results predict in fact a subdiffusive spreading of energy slower than any power law.

5.2   Disordered classical spin chain

In [23], a one-dimensional chain of classical spins is studied. Let $\mathbb{S}$ be the unit sphere in $\mathbb{R}^3$, and let a point of the phase space be given by $\mathbf{S} = (\mathbf{S}_x)_{x \in \mathbb{Z}_N} \in \mathbb{S}^N$. The hamiltonian H is given by

$$H(\mathbf{S}) \;=\; \sum_{x \in \mathbb{Z}_N} \boldsymbol{\omega}_x \cdot \mathbf{S}_x + \epsilon \sum_{x \in \mathbb{Z}_N} \mathbf{S}_x \cdot \mathbf{S}_{x+1},$$

and the equations of motions read

$$\dot{\mathbf{S}}_x \;=\; \nabla_x H \wedge \mathbf{S}_x, \tag{21}$$

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[3] While this looks reasonable, it is fair to say that this identification is not justified, since we do not look at the system in a true macroscopic scale.

where $\nabla_x = (\partial/\partial S_{1,x}, \partial/\partial S_{2,x}, \partial/\partial S_{3,x})$, where $\mathbf{S} \cdot \mathbf{T}$ is the standard scalar product in $\mathbb{R}^3$, and where $\mathbf{S} \wedge \mathbf{T}$ is the standard vector product in $\mathbb{R}^3$. It is assumed that $(\boldsymbol{\omega}_x)_{x \in \mathbb{Z}_N}$ form a sequence of independent and identically distributed random vectors in $\mathbb{R}^3$.

The dynamics (21) is generated by A, defined as

$$Au = \sum_{x \in \mathbb{Z}_N} (\nabla_x H \wedge \mathbf{S}_x) \cdot \nabla_x u = \sum_{x \in \mathbb{Z}_N} (\boldsymbol{\omega}_x \wedge \mathbf{S}_x) \cdot \nabla_x u + \epsilon \sum_{x \in \mathbb{Z}_N} ((\mathbf{S}_{x-1} + \mathbf{S}_{x+1}) \wedge \mathbf{S}_x) \cdot \nabla_x u$$

$$= A_{har} u + \epsilon A_{anh} u$$

where the denominations A_{har} and A_{anh} are aimed to bare the analogy with our model. The energy of the particle at site x is defined as

$$e_x = \boldsymbol{\omega}_x \cdot \mathbf{S}_x + \frac{\epsilon}{2} \mathbf{S}_{x-1} \cdot \mathbf{S}_x + \frac{\epsilon}{2} \mathbf{S}_x \cdot \mathbf{S}_{x+1},$$

and the relation

$$Ae_x = \epsilon(j_{x-1,x} - j_{x,x+1})$$

is satisfied if one defines the current $j_{x,x+1}$ by

$$j_{x,x+1} = \frac{\boldsymbol{\omega}_x + \boldsymbol{\omega}_{x+1}}{2} \cdot (\mathbf{S}_x \wedge \mathbf{S}_{x+1}) + \frac{\epsilon}{2} ((\mathbf{S}_{x-1} \wedge \mathbf{S}_x) \cdot \mathbf{S}_{x+1} + (\mathbf{S}_x \wedge \mathbf{S}_{x+1}) \cdot \mathbf{S}_{x+2}).$$

At $\epsilon = 0$, each spin just precesses around the axis characterized by the vector $\boldsymbol{\omega}_x$, at a constant angular velocity $|\boldsymbol{\omega}_x|_2$. Its energy $\boldsymbol{\omega}_x \cdot \mathbf{S}_x$ is conserved, and determines the plane perpendicular to $\boldsymbol{\omega}_x$ where the precession takes place. The picture in phase space is thus here completely analogous to that of the one-dimensional harmonic oscillators we have considered.

To proceed as before, we would now need to partition the set of functions on the phase space into two subspaces, playing the role of p-symmetric and p-antisymmetric functions. This is easily made possible if we add the extra assumption that, writing $\boldsymbol{\omega}_x = (\omega_{1,x}, \omega_{2,x}, \omega_{3,x})$, one has almost surely

$$\omega_{1,x} = 0 \qquad \text{for all } x \in \mathbb{Z}_N. \tag{22}$$

We then define 1-symmetric functions on $\mathbb{S}^N$ as functions that are symmetric under the reflection that simultaneously maps $\omega_{1,x}$ to $-\omega_{1,x}$ for every $x \in \mathbb{Z}_N$; 1-antisymmetric functions are defined similarly. Then the currents $j_{x,x+1}$ are 1-antisymmetric functions, and the generator A maps 1-symmetric functions to 1-antisymmetric functions and vice versa. This is all what is needed to compute a finite number of perturbative steps as we did. Without the assumption (22), we have not been able to find a symmetry that guarantees the solvability of the hierarchy (8)-(9). Still, such a symmetry is noway a necessary condition ; we believe that this is only a technical question, and that these equations can in fact be solved.

5.3 One dimensional chain of strongly anharmonic oscillators

We finally suggest an analogy with a a one-dimensional chain of identical strongly anharmonic oscillators. The phase space is the set of points $(q, p) \in \mathbb{R}^{2N}$, and the hamiltonian is given by

$$H(q, p) = \sum_{x \in \mathbb{Z}_N} \left(\frac{p_x^2}{2} + \frac{q_x^4}{4} \right) + \frac{\epsilon}{2} \sum_{x \in \mathbb{Z}_N} (q_x - q_{x+1})^2.$$

Equations of motion are the usual Hamilton equations. For such a system, existence of breathers is well known in the infinite volume limit [21]. Unfortunately, this as such does not say us much about the conductivity of the chain.

Let us see that, at least in the asymptotic regime we consider, the effect of the strongly anharmonic onsite potentials could be compared to that of random frequencies. At $\epsilon = 0$, all the oscillators are characterized by an energy $e_{x,0}$, and evolve independently from each other. Since their dynamics is one-dimensional, they are just oscillating periodically at a frequency $\omega_{x,0} \sim (e_{x,0})^{1/4}$. So, as far as the uncoupled dynamics is concerned, a typical state of a chain of identical anharmonic oscillators on the one hand, and a typical state of a disordered chain of harmonic oscillators on the other hand, are qualitatively completely similar. In the first case however, typical means typical with respect to the Gibbs measure, while in the second it means typical with respect to both the Gibbs measure and the disorder.

However, when trying to adapt our strategy, the problem of resonances cannot be ruled out so easily. Indeed, in the disordered chain, resonances only affected some fixed sites, while here, the places where resonances occur also move with time, a situation which in fact favors the transport of energy. Still our main intuition remains that, due to quick averaging of fast oscillations, energy gets trapped in finite portions of space for time-scales that might not be an inverse polynomial of ϵ.

References

[1] O. Ajanki, F. Huveneers, Rigorous Scaling Law for the Heat Current in Disordered Harmonic Chain, Communication in Mathematical Physics, 301 (3), 841-883, 2011.

[2] K. Aoki, J. Lukkarinen, H. Spohn, Energy Transport in Weakly Anharmonic Chains, Journal of Statistical Physics, 124 (5), 1105-1129, 2006.

[3] D. M. Basko, Weak chaos in the disordered nonlinear Schrödinger chain: Destruction of Anderson localization by Arnold diffusion, Annals of Physics, 326 (7), 1577-1655, 2011.

[4] G. Benabou, Homogenization of Ornstein-Uhlenbeck Process in Random Environment, Communications in Mathematical Physics, 266 (3), 699-714, 2006.

[5] C. Bernardin, Thermal Conductivity for a Noisy Disordered Harmonic Chain, Journal of Statistical Physics, 133 (3), 417-433, 2008.

[6] C. Bernardin, F. Huveneers, Small perturbation of a disordered harmonic chain by a noise and an anharmonic potential, Probability Theory and Related Fields, Online First, DOI 10.1007/s00440-012-0458-8, 1-31, 2012.

[7] C. Bernardin, S. Olla, Transport Properties of a Chain of Anharmonic Oscillators with Random Flip of Velocities, Journal of Statistical Physics, 145 (5), 1224-1255, 2011.

[8] T. Bodineau, B. Helffer, Correlations, Spectral Gap and Log-Sobolev inequalities for unbounded spin systems, pp. 51-66 in Differential Equations and Mathematical Physics: Proceedings of an International Conference Held at the University of Alabama in Birmingham, March 16-20, 1999, edited by R. Weikard, G. Weinstein, American Mathematical Society and International Press, 2000.

[9] F. Bonetto, J.L. Lebowitz, L. Rey-Bellet, Fouriers law : a challenge to theorists, pp. 128-150 in Mathematical physics 2000, edited by A. Fokas, A. Grigoryan, T. Kibble and B. Zegarlinsky, Imperial College Press, 2000.

[10] A. Kundu, A. Chaudhuri, D. Roy, A. Dhar, J. L. Lebowitz, H. Spohn, Heat transport and phonon localization in mass-disordered harmonic crystals Physical Review B 81 (6), 064301 (17 pages), 2010.

[11] A. Casher, J. L. Lebowitz, Heat Flow in Regular and Disordered Harmonic Chains, Journal of Mathematical Physics, 12 (8), 1701-1711, 1971.

[12] A. Dhar, J. L. Lebowitz, Effect of Phonon-Phonon Interactions on Localization, Physical Review Letters, 100 (13), 134301 (4 pages), 2008.

[13] D. Dolgopyat, C. Liverani, Energy Transfer in a Fast-Slow Hamiltonian System, Communications in Mathematical Physics, 308 (1), 201-225, 2011.

[14] J. Fröhlich, T. Spencer, C. E. Wayne, Localization in Disordered, Nonlinear Dynamical Systems, Journal of Statistical Physics, 42 (3-4), 247-274, 1986.

[15] R Lefevere, A. Schenkel, Normal heat conductivity in a strongly pinned chain of anharmonic oscillators, Journal of Statistical Mechanics: Theory and Experiment, L02001 (9 pages), 2006.

[16] M. Hairer, J. C. Mattingly, Slow energy dissipation in anharmonic oscillator chains, Communications on Pure and Applied Mathematics, 62 (8), 999-1032, 2009.

[17] C. Kipnis, C. Landim, Scaling Limits of Interacting Particle Systems, Springer-Verlag, Berlin, 1999.

[18] S. Lepri, R. Livi, A. Politi, Thermal conduction in classical low-dimensional lattices, Physics Reports 377, 1-80, 2003.

[19] C. Liverani, S. Olla, Toward the Fourier law for a weakly interacting anharmonic crystal, Journal of the American Mathematical Society, 25 (2), 555-583, 2012.

[20] J. Lukkarinen, H. Spohn Anomalous energy transport in the FPU-β chain, Communications on Pure and Applied Mathematics, 61 (12), 1753-1786, 2008.

[21] R. S. MacKay, S. Aubry, Proof of existence of breathers for time-reversible or Hamiltonian networks of weakly coupled oscillators, Nonlinearity, 7 (6), 1623-1643, 1994.

[22] M. Mulansky, A. Pikovsky, Spreading in disordered lattices with different nonlinearities, Europhysics Letters, 90, 10015 (6 pages), 2010.

[23] V. Oganesyan, A. Pal, D. Huse, Energy transport in disordered classical spin chains, Physical Review B, 80 (11), 115104 (8 pages), 2009.

[24] J. Pöschel, Small Divisors with Spatial Structure in Infinite Dimensional Hamiltonian Systems, Communications in Mathematical Physics, 127 (2), 351-393, 1990.

[25] J. Pöschel, A lecture on the classical KAM theorem, Proceedings of Symposia in Pure Mathematics, 69, 707-732, 2001.

[26] Z. Rieder, J.L. Lebowitz, E.H. Lieb, Properties of Harmonic Crystal in a Stationary Nonequilibrium State, Journal of Mathematical Physics, 8 (5), 1073-1078, 1967.

[27] R. J. Rubin, W. L. Greer, Abnormal Lattice Thermal Conductivity of a One-Dimensional, Harmonic, Isotopically Disordered Crystal, Journal of Mathematical Physics, 12 (8), 1686-1701, 1971.

[28] B. Ryals, L.-S. Young, Nonequilibrium Steady States of Some Simple 1-D Mechanical Chains, Journal of Statistical Physics, 146 (5), 1089-1103, 2012.

[29] T. Verheggen, Transmission coefficient and heat conduction of a harmonic chain with random masses: Asymptotic estimates on products of random matrices, Communications in Mathematical Physics, 68 (1), 69-82, 1979.